A NEW CHARACTERIZATION OF THE BERGER SPHERE IN COMPLEX PROJECTIVE SPACE

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Abstract. We give a complete classification of Lagrangian immersions of homogeneous 3-manifolds (the Berger spheres, the Heisenberg group Nil_3, the universal covering of the Lie group $\mathrm{PSL}(2,\mathbb{R})$ and the Lie group Sol_3) in 3-dimensional complex space forms. As a corollary, we get a new characterization of the Berger sphere in complex projective space.

1. Introduction

Let $\phi : M^n \to \bar{M}^n$ be an isometric immersion from an n-dimensional Riemannian manifold into a complex n-dimensional Kähler manifold $\bar{M}^n$. M^n is called a *Lagrangian submanifold* if the almost complex structure J of $\bar{M}^n$ carries each tangent space of M^n into its corresponding normal space.

In this paper we study Lagrangian submanifolds of the 3-dimensional complex space forms. In the last decades many results about Lagrangian submanifolds have been obtained characterizing several classes of submanifolds. Some are valid in all dimensions, while others are only valid in dimension 2 or 3. For a recent survey, we refer to the books of B.-Y. Chen (see [2] and [5]). Nevertheless, even in low dimensions, many open problems remain.

On the other hand, also in the last decades, many people have started to study the geometry of some special 3-dimensional homogeneous spaces. In particular many works have been devoted to study surfaces in such manifolds, see among others the very nice survey by Fernández and Mira [15]. In the simply connected case, the classification of 3-dimensional homogeneous manifolds is well-known. Such manifolds can be divided into several classes depending on the dimension of the isometry group. They include

(1) the spaces with 6-dimensional isometry group (Euclidean space $\mathbb{R}^3$, standard sphere $\mathbb{S}^3$ and hyperbolic space $\mathbb{H}^3$).
(2) the spaces with 4-dimensional isometry group (which consist of $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, the Berger spheres, the Heisenberg group Nil_3 and the universal covering of the Lie group $\mathrm{PSL}(2,\mathbb{R})$).
(3) the spaces with 3-dimensional isometry group (which are a certain class of Lie groups, of which the Lie group Sol_3 is specially important, since it is the only Thurston geometry among them).

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From another point of view, equally interesting questions are whether any of these 3-dimensional homogeneous spaces can be isometrically immersed (locally or globally) as a Lagrangian submanifold in a 3-dimensional complex space form, and whether such immersions (if they exist) are necessarily unique. In this paper, we will investigate the Lagrangian immersions of homogeneous 3-manifolds in complex space forms.

A first result which can be formulated within this framework is the well known result by Ejiri ([13]).

Theorem 1.1 ([13])**.** *Let ϕ be a minimal Lagrangian immersion of a 3-dimensional manifold with constant sectional curvature in a 3-dimensional complex space form. Then either*

- *M^3 is totally geodesic, or*
- *M^3 is congruent to a flat torus in $\mathbb{CP}^3$.*

Note that without the assumption of minimality, as far as the authors know, the answer is not known. Even assuming that the immersion is a global one. Some partial results with respect to this question were obtained in [21] and [7]. In this paper, we will be interested in the existence and uniqueness of Lagrangian immersions of the homogeneous 3-manifolds (the Berger spheres, the Heisenberg group Nil_3, the universal covering of the Lie group $\mathrm{PSL}(2,\mathbb{R})$, the Lie group Sol_3, the product spaces $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$) in 3-dimensional complex space forms. Note that all these spaces have a common property, that is, the Ricci tensor has exactly two different eigenvalues at every point, this property will play an important role in the proof of our main results.

We will first show that, in contrast to Lagrangian immersions of real space forms in complex space forms, a Lagrangian isometric immersion (even locally) of one of the following homogeneous 3-manifolds (the Berger spheres, the Heisenberg group Nil_3, the universal covering of the Lie group $\mathrm{PSL}(2,\mathbb{R})$ and the Lie group Sol_3) in a 3-dimensional complex space form is always minimal. Moreover the only possibility is a unique isometric Lagrangian immersion of the Berger sphere $\mathbb{S}^3_b(4/3,1)$ in $\mathbb{CP}^3(4)$. As a corollary, we get a new characterization of the Berger sphere in complex projective space.

Theorem 1.2. *Let ϕ be a Lagrangian isometric immersion from (an open part of) one of the homogeneous 3-manifolds M^3 (the Berger spheres, the Heisenberg group Nil_3, the universal covering of the Lie group $PSL(2,\mathbb{R})$ and the Lie group Sol_3) to a 3-dimensional complex space form $\bar{M}^3(4c), c\in\{-1,0,1\}$. Then $c=1$, ϕ is minimal and M^3 is the Berger sphere $\mathbb{S}^3_b(4/3,1)$. And up to an isometry of $\mathbb{CP}^3(4)$, the immersion ϕ is unique.*

Remark 1.3. In Theorem 1.2, for simplicity, we assume that $c\in\{-1,0,1\}$. If we do not assume this, we get that $c>0$, ϕ is minimal and M^3 is a Berger sphere $\mathbb{S}^3_b(4c/3,c)$. And up to an isometry of $\mathbb{CP}^3(4c)$, the immersion ϕ is unique.

In the proof of Theorem 1.2, one of the key ingredients is trying to choose a good local frame such that the second fundamental form has a simple form under this frame. To achieve this, we first deal with a more general case. We consider M^3 to be a quasi-Einstein manifold immersed in a complex space form $\bar{M}^3(4c)$. Here quasi-Einstein means that the Ricci tensor of M^3 has an eigenvalue of multiplicity at least 2 at every point, hence quasi-Einstein manifold is a generalization of Einstein manifold. If M^3 is quasi-Einstein and not Einstein, then at every point of M^3 the Ricci tensor Ric has exactly two different eigenvalues. In this case, we find a very nice property of M^3, that is, at each point p of M^3, if e_3 is a unit eigenvector corresponding to the simple eigenvalue of Ric, e_1 and e_2 are arbitrary orthonormal vectors belonging to the 2-dimensional eigenspace of Ric, then e_1, e_2 and e_3 are also eigenvectors of A_{Je_3}, with e_1 and e_2 belonging to the same eigenspace of A_{Je_3}. Since $\{e_1, e_2, e_3\}$ are eigenvectors of Ric, we can always find local vector fields $\{E_1, E_2, E_3\}$ such that at each point, E_1, E_2 and E_3 are eigenvectors of Ric, with E_3 corresponding to the simple eigenvalue of Ric. Thus under the local vector fields $\{E_1, E_2, E_3\}$, the second fundamental form has a simple form. This will be done in Proposition 4.1, which is the key step in the proof of our main results.

When the immersion is moreover minimal, we get Corollary 4.2 from Proposition 4.1, we find a better local frame such that the second fundamental form has an easier form. Using Corollary 4.2, we prove the following classification theorem for minimal Lagrangian quasi-Einstein submanifolds in a complex space form $\bar{M}^3(4c)$.

Theorem 1.4. *Let M^3 be a minimal Lagrangian quasi-Einstein submanifold in a complex space form $\bar{M}^3(4c)$, $c \in \{-1, 0, 1\}$. Assume that M^3 has constant scalar curvature, then locally M^3 is congruent to one of the following*

- *M^3 is totally geodesic.*
- *M^3 is a flat torus in $\mathbb{CP}^3$.*
- *M^3 is the Berger sphere $\mathbb{S}^3_b(\frac{4}{3}, 1)$ with the immersion given by $\phi : \mathbb{S}^3_b(\frac{4}{3}, 1) \to \mathbb{CP}^3(4)$, $(z, w) \mapsto [\bar{z}^3 + 3\bar{z}\bar{w}^2, \sqrt{3}(\bar{z}^2 w + \bar{w}|w|^2 - 2\bar{w}|z|^2), \sqrt{3}(\bar{z}w^2 + z|z|^2 - 2z|w|^2), w^3 + 3z^2 w]$.*
- *M^3 is locally isometric with $\mathbb{S}^2 \times \mathbb{R}$, $\bar{M}^3(4c) = \mathbb{CP}^3$, and the immersion is obtained as the Calabi product of the totally geodesic immersion of $\mathbb{S}^2$ into $\mathbb{CP}^2$ and a point.*

Remark 1.5. Since quasi-Einstein manifold is a generalization of Einstein manifold, Theorem 1.4 can be regarded as a generalization of Ejiri's result (see Theorem 1.1).

Finally, for other two types of homogeneous 3-manifolds (the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$), in order to be able to reach a conclusion, as in the constant sectional curvature case, we assume that the immersion is minimal. In this case, we have

Theorem 1.6. *Let ϕ be a minimal Lagrangian isometric immersion from (an open part of) $\mathbb{S}^2 \times \mathbb{R}$ to a 3-dimensional complex space form $\bar{M}^3(4c), c \in \{-1, 0, 1\}$. Then $c = 1$, ϕ is obtained as the Calabi product of the totally geodesic immersion of $\mathbb{S}^2$ into $\mathbb{CP}^2$ and a point.*

Theorem 1.7. *There exists no minimal Lagrangian isometric immersion from an open part of $\mathbb{H}^2 \times \mathbb{R}$ into a 3-dimensional complex space form $\bar{M}^3(4c), c \in \{-1, 0, 1\}$.*

Remark 1.8. Notice that both $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are quasi-Einstein and have constant scalar curvatures, hence Theorems 1.6 and 1.7 could be viewed as direct consequences of Theorem 1.4.

The paper is organized as follows. In section 2, we recall the basic formulas for Lagrangian submanifolds of complex space forms. In section 3, we review some basic properties of the homogeneous 3-manifolds with isometry group of dimension 4 or 3. In section 4, we first prove Proposition 4.1, which is the key step in the proof of our main results. Finally, we prove Theorems 1.2, 1.4, 1.6 and 1.7.

2. Preliminaries

In this section, M will always denote an n-dimensional Lagrangian submanifold of $\bar{M}^n(4c)$, which is an n-dimensional complex space form with constant holomorphic sectional curvature $4c$. We denote the Levi-Civita connections on M, $\bar{M}^n(4c)$ and the normal bundle by ∇, D and $\nabla^{\perp}_X$ respectively. The formulas of Gauss and Weingarten are given by (see [1], [2] and [8])

$$D_X Y = \nabla_X Y + h(X,Y),\ D_X\xi = -A_\xi X + \nabla^{\perp}_X \xi,$$

where X and Y are tangent vector fields and ξ is a normal vector field on M.

The Lagrangian condition implies that

$$\nabla^{\perp}_X JY = J\nabla_X Y,\ A_{JX}Y = -Jh(X,Y) = A_{JY}X,$$

where h is the second fundamental form and A denotes the shape operator.

We denote the curvature tensors of ∇ and $\nabla^{\perp}_X$ by R and $R^{\perp}$ respectively. The first covariant derivative of h is defined by

$$(\nabla h)(X,Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \tag{2.1}$$

where X, Y, Z and W are tangent vector fields.

The equations of Gauss, Codazzi and Ricci for a Lagrangian submanifold of $\bar{M}^n(4c)$ are given by (see [3] and [4])

$$\begin{aligned}\langle R(X,Y)Z,W\rangle\ &= \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle \\ &+ c\ (\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle),\end{aligned} \tag{2.2}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.3}$$

$$\begin{aligned}\langle R^{\perp}(X,Y)JZ, JW\rangle\ &= \langle [A_{JZ}, A_{JW}]X, Y\rangle \\ &+ c\ (\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle),\end{aligned} \tag{2.4}$$

where X, Y Z and W are tangent vector fields. Note that for a Lagrangian submanifold the equations of Gauss and Ricci are mutually equivalent.

The Lagrangian condition implies that

$$\langle R^{\perp}(X,Y)JZ, JW\rangle\ =\ \langle R(X,Y)Z,W\rangle, \tag{2.5}$$

$$\langle h(X,Y), JZ\rangle\ =\ \langle h(X,Z), JY\rangle, \tag{2.6}$$

for tangent vector fields X, Y, Z and W. From (2.3) and (2.6), we also have

$$\langle (\nabla h)(W,X,Y), JZ\rangle\ =\ \langle (\nabla h)(W,X,Z), JY\rangle, \tag{2.7}$$

for tangent vector fields X, Y, Z and W.

3. Homogeneous 3-manifolds with isometry group of dimension 4 or 3

In this section, we review some basic properties of the homogeneous 3-manifolds with isometry group of dimension 4 or 3.

3.1. **Homogeneous 3-manifolds with isometry group of dimension 4.**

Any simply connected homogeneous 3-manifold with 4-dimensional isometry group admits (see [15] and [9]) a fibration over $M^2(\kappa)$ (a 2-dimensional space form of constant sectional curvature κ), for some $\kappa \in \mathbb{R}$. These manifolds can be parameterized in terms of the base curvature κ and the bundle curvature τ, that satisfying $\kappa - 4\tau^2 \neq 0$. We will use the notation $\mathbb{E}^3(\kappa,\tau)$ for these homogeneous 3-manifolds.

- When $\tau = 0$, the corresponding spaces are the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.
- When $\tau \neq 0$ and $\kappa > 0$, we obtain the Berger spheres.
- When $\tau \neq 0$ and $\kappa = 0$, the corresponding space is the Heisenberg group Nil_3.
- When $\tau \neq 0$ and $\kappa < 0$, we obtain the universal covering of the Lie group $\mathrm{PSL}(2,\mathbb{R})$.

The Berger spheres and the exotic Lagrangian immersion of $\mathbb{S}^3$ in $\mathbb{CP}^3$. A Berger sphere is a usual 3-dimensional sphere $\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ endowed with the metric (see [23])

$$g(X,Y) = \frac{4}{\kappa}\left[\langle X,Y\rangle + \left(\frac{4\tau^2}{\kappa} - 1\right)\langle X,V\rangle\langle Y,V\rangle\right],$$

where $\langle\, ,\, \rangle$ stands for the usual metric on the sphere, $V_{(z,w)} = (iz, iw)$, for each $(z,w) \in \mathbb{S}^3$ and κ, τ are real numbers with $\kappa > 0$, $\tau \neq 0$ and $\kappa \neq 4\tau^2$. From now on we will denote the Berger sphere $(\mathbb{S}^3, g)$ by $\mathbb{S}^3_b(\kappa,\tau)$.

On a Berger sphere $\mathbb{S}^3_b(\kappa,\tau)$, there exist three natural orthonormal vector fields

$$E_1 = \frac{\sqrt{\kappa}}{2}(-\bar{w}, \bar{z}),\ E_2 = \frac{\sqrt{\kappa}}{2}(-i\bar{w}, i\bar{z}), E_3 = \frac{\kappa}{4\tau}(iz, iw). \tag{3.1}$$

The Levi-Civita connections are given by

$$\begin{aligned}
&\nabla_{E_i}E_i = 0,\ i = 1,2,3,\\
&\nabla_{E_1}E_2 = -\tau E_3,\ \nabla_{E_1}E_3 = \tau E_2,\\
&\nabla_{E_2}E_1 = \tau E_3,\ \nabla_{E_2}E_3 = -\tau E_1,\\
&\nabla_{E_3}E_1 = (\tau - \frac{\kappa}{2\tau})E_2,\ \nabla_{E_3}E_2 = (\frac{\kappa}{2\tau} - \tau)E_1.
\end{aligned} \tag{3.2}$$

The Riemannian curvature tensor R of $\mathbb{S}^3_b(\kappa,\tau)$ is given by

$$\begin{aligned}
&R(E_1,E_2)E_2 = (\kappa - 3\tau^2)E_1,\ R(E_1,E_2)E_3 = 0,\\
&R(E_1,E_3)E_3 = \tau^2 E_1,\ R(E_1,E_3)E_2 = 0,\\
&R(E_2,E_3)E_3 = \tau^2 E_2,\ R(E_2,E_3)E_1 = 0.
\end{aligned} \tag{3.3}$$

The Ricci curvature is given by

$$R_{11} = R_{22} = \kappa - 2\tau^2, R_{33} = 2\tau^2,\ R_{ij} = 0, i \neq j. \tag{3.4}$$

It was shown in [6] that there exists an exotic minimal Lagrangian immersion of the topological $\mathbb{S}^3$ in $\mathbb{CP}^3$. From a geometric point of view, this is a Lagrangian isometric immersion of the Berger sphere $\mathbb{S}^3_b(\frac{4}{3}, 1)$ into $\mathbb{CP}^3(4)$, with the second fundamental form satisfying that

$$\begin{aligned}
&h(E_1,E_1) = \frac{2}{\sqrt{3}}JE_1,\ h(E_2,E_2) = -\frac{2}{\sqrt{3}}JE_1,\\
&h(E_1,E_2) = -\frac{2}{\sqrt{3}}JE_2,\ h(E_2,E_3) = 0,\\
&h(E_1,E_3) = 0,\ h(E_3,E_3) = 0.
\end{aligned} \tag{3.5}$$

We mention that the above example has an analytic expression, which is given by $\phi : \mathbb{S}^3_b(\frac{4}{3}, 1) \to \mathbb{CP}^3(4)$, $(z,w) \mapsto [\bar{z}^3 + 3\bar{z}\bar{w}^2, \sqrt{3}(\bar{z}^2 w + \bar{w}|w|^2 - 2\bar{w}|z|^2), \sqrt{3}(\bar{z}w^2 + z|z|^2 - 2z|w|^2), w^3 + 3z^2 w]$. For more details on how to get the analytic expression, see [18] or [14].

The Heisenberg group Nil_3. The Heisenberg group is the Lie group (see [9])

$$\mathrm{Nil}_3 = \{\begin{bmatrix} 1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}; (x,y,z) \in \mathbb{R}^3\}$$

endowed with a left invariant metric:

$$ds^2 = dx^2 + dy^2 + (\tau(ydx - xdy) + dz)^2.$$

The canonical frame $\{E_1, E_2, E_3\}$ is given by

$$E_1 = \partial_x - \tau y\partial_z, E_2 = \partial_y + \tau x\partial_z, E_3 = \partial_z.$$

The Levi-Civita Riemannian connection is determined by

$$\begin{aligned}
&\nabla_{E_i}E_i = 0,\ i = 1, 2, 3,\\
&\nabla_{E_1}E_2 = \tau E_3,\ \nabla_{E_1}E_3 = -\tau E_2,\\
&\nabla_{E_2}E_1 = -\tau E_3,\ \nabla_{E_2}E_3 = \tau E_1,\\
&\nabla_{E_3}E_1 = -\tau E_2,\ \nabla_{E_3}E_2 = \tau E_1.
\end{aligned} \tag{3.6}$$

The Riemannian curvature tensor R of (Nil_3, ds^2) is given by

$$\begin{aligned}
&R(E_1, E_2)E_2 = -3\tau^2E_1,\ R(E_1, E_2)E_3 = 0,\\
&R(E_1, E_3)E_3 = \tau^2E_1,\ R(E_1, E_3)E_2 = 0,\\
&R(E_2, E_3)E_3 = \tau^2E_2,\ R(E_2, E_3)E_1 = 0.
\end{aligned} \tag{3.7}$$

The Ricci curvature is given by

$$R_{11} = R_{22} = -2\tau^2, R_{33} = 2\tau^2,\ R_{ij} = 0, i \neq j. \tag{3.8}$$

The universal covering of the Lie group PSL$(2,\mathbb{R})$: $\widetilde{\mathbf{PSL}(2,\mathbb{R})}$. We use the description in [22]. It is the Lie group

$$\widetilde{\mathrm{PSL}(2,\mathbb{R})} = \{(z, w) \in \mathbb{C}^2; |z|^2 - |w|^2 = 1\}$$

endowed with the metric

$$g(F_i, F_j) = \delta_{ij}\frac{4}{-\kappa}, g(V, V) = \frac{16\tau^2}{\kappa^2}, g(V, F_j) = 0, i, j = 1, 2,$$

where κ and τ are real numbers such that $\kappa < 0, \tau \neq 0$, and $\{F_1, F_2, V\}$ is a global frame defined by

$$F_1 = (\bar{w}, \bar{z}),\ F_2 = (i\bar{w}, i\bar{z}), V = (iz, iw).$$

Set

$$E_1 = \frac{\sqrt{-\kappa}}{2}(\bar{w}, \bar{z}),\ E_2 = \frac{\sqrt{-\kappa}}{2}(i\bar{w}, i\bar{z}), E_3 = \frac{\kappa}{4\tau}(iz, iw), \tag{3.9}$$

we get a canonical frame on $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$. The Levi-Civita connections are given by

$$\begin{aligned}
&\nabla_{E_i}E_i = 0,\ i = 1, 2, 3,\\
&\nabla_{E_1}E_2 = -\tau E_3,\ \nabla_{E_1}E_3 = \tau E_2,\\
&\nabla_{E_2}E_1 = \tau E_3,\ \nabla_{E_2}E_3 = -\tau E_1,\\
&\nabla_{E_3}E_1 = (\tau - \frac{\kappa}{2\tau})E_2,\ \nabla_{E_3}E_2 = (\frac{\kappa}{2\tau} - \tau)E_1.
\end{aligned} \tag{3.10}$$

The Riemannian curvature tensor R of $(\widetilde{\mathrm{PSL}(2,\mathbb{R})}, g)$ is given by

$$\begin{aligned}
&R(E_1, E_2)E_2 = (\kappa - 3\tau^2)E_1,\ R(E_1, E_2)E_3 = 0,\\
&R(E_1, E_3)E_3 = \tau^2E_1,\ R(E_1, E_3)E_2 = 0,\\
&R(E_2, E_3)E_3 = \tau^2E_2,\ R(E_2, E_3)E_1 = 0.
\end{aligned} \tag{3.11}$$

The Ricci curvature is given by

$$R_{11} = R_{22} = \kappa - 2\tau^2, R_{33} = 2\tau^2,\ R_{ij} = 0, i \neq j. \tag{3.12}$$

3.2. **Homogeneous 3-manifolds with isometry group of dimension 3.** The Lie group Sol_3 is the only Thurston geometry among all homogeneous 3-manifolds with 3-dimensional isometry group. We will use the descriptions in [10]. The Lie group Sol_3 can be viewed as $\mathbb{R}^3$ endowed with the Riemannian metric

$$ds^2 = e^{2x_3}dx_1^2 + e^{-2x_3}dx_2^2 + dx_3^2.$$

The canonical frame is defined by

$$E_1 = e^{-x_3}\partial_{x_1}, E_2 = e^{x_3}\partial_{x_2}, E_3 = \partial_{x_3}.$$

The Levi-Civita connections are given by

$$\begin{aligned} &\nabla_{E_1}E_1 = -E_3,\ \nabla_{E_2}E_1 = 0,\ \nabla_{E_3}E_1 = 0,\\ &\nabla_{E_1}E_2 = 0,\ \nabla_{E_2}E_2 = E_3,\ \nabla_{E_3}E_2 = 0,\\ &\nabla_{E_1}E_3 = E_1,\ \nabla_{E_2}E_3 = -E_2,\ \nabla_{E_3}E_3 = 0. \end{aligned} \tag{3.13}$$

The Riemannian curvature tensor R of (Sol_3, ds^2) is given by

$$\begin{aligned} &R(E_1,E_2)E_2 = E_1,\ R(E_1,E_2)E_3 = 0,\\ &R(E_1,E_3)E_3 = -E_1,\ R(E_1,E_3)E_2 = 0,\\ &R(E_2,E_3)E_3 = -E_2,\ R(E_2,E_3)E_1 = 0. \end{aligned} \tag{3.14}$$

The Ricci curvature is given by

$$R_{11} = R_{22} = 0, R_{33} = -2,\ R_{ij} = 0, i \neq j. \tag{3.15}$$

4. Proof of Theorems 1.2, 1.4, 1.6 and 1.7

We start by dealing with a more general case. We assume that M^3 is quasi-Einstein (i.e. the Ricci tensor has an eigenvalue of multiplicity at least 2 at every point). If M^3 is Einstein, since the dimension of M^3 is 3, by Schur Lemma, it follows that M^3 has constant sectional curvature. As this is excluded by the assumptions of our theorems, we will mainly consider the case when M^3 is not Einstein. By restricting, if necessary to an open dense subset of M^3, this means that we may assume that the Ricci tensor of M^3 has exactly two different eigenvalues at every point p.

First, we prove the following key proposition, which is the most important step in the proof of our main results. The idea to prove Proposition 4.1 is to exploit as more information as possible from the property that the Ricci tensor has exactly two different eigenvalues.

Proposition 4.1. *Let M^3 be a Lagrangian quasi-Einstein submanifold in a complex space form $\bar{M}^3(4c)$, $c \in \{-1,0,1\}$. Assume that M^3 is not Einstein. Let $p \in M$ be a non-totally geodesic point, then there exists a neighborhood U of p such that the Ricci tensor Ric of M^3 has exactly two different eigenvalues at each point of U. Under this assumption, we have that there exists an open subset V of U and local orthonormal vector fields $\{E_1, E_2, E_3\}$ such that at each point of V, E_1, E_2, E_3 are eigenvectors of Ric, with E_1 and E_2 belonging to the 2-dimensional eigenspace and E_3 belonging to the 1-dimensional eigenspace. Moreover, at each point of V, E_1, E_2, E_3 are eigenvectors of A_{JE_3}, with E_1 and E_2 belonging to the same eigenspace of A_{JE_3}. And there exist local functions $a_1, a_2, b_1, b_2, b_3, c_3$ such that at each point of V, we have that*

$$\begin{cases} h(E_1,E_1) = a_1JE_1 + a_2JE_2 + b_3JE_3,\ h(E_2,E_2) = b_1JE_1 + b_2JE_2 + b_3JE_3,\\ h(E_1,E_2) = a_2JE_1 + b_1JE_2,\ h(E_2,E_3) = b_3JE_2,\\ h(E_1,E_3) = b_3JE_1,\ h(E_3,E_3) = c_3JE_3. \end{cases} \tag{4.1}$$

Moreover, if we change E_1 and E_2 in the 2-dimensional eigenspace of the Ricci tensor, the second fundamental form has the same form.

Proof. Since p is not a totally geodesic point, we can choose $e_1 \in T_pM$ such that $f(v) = \langle h(v,v), Jv\rangle,\ |v| = 1$ attains its maximum at e_1. Hence $\langle h(e_1,e_1), Jv\rangle = 0,\ \forall\ v \perp e_1$. Thus we can choose an orthonormal basis $\{e_2, e_3\}$ in the space which is orthogonal to e_1 such that e_2 and e_3 are eigenvectors of A_{Je_1}. Under this basis $\{e_1, e_2, e_3\}$, the second fundamental form has the following form.

$$\begin{aligned} &h(e_1,e_1) = a_1 Je_1,\ h(e_2,e_2) = b_1 Je_1 + b_2 Je_2 + b_3 Je_3,\\ &h(e_1,e_2) = b_1 Je_2,\ h(e_2,e_3) = b_3 Je_2 + c_2 Je_3,\\ &h(e_1,e_3) = c_1 Je_3,\ h(e_3,e_3) = c_1 Je_1 + c_2 Je_2 + c_3 Je_3. \end{aligned} \tag{4.2}$$

Using the Gauss equation (2.2), we get the matrix form of the Ricci curvature.

$$(R_{ij}) = \begin{bmatrix} \lambda_1 & \mu_1 & \mu_2 \\ \mu_1 & \lambda_2 & 0 \\ \mu_2 & 0 & \lambda_3 \end{bmatrix},$$

where

$$\begin{aligned} &\lambda_1 = 2c - b_1^2 - c_1^2 + a_1(b_1 + c_1), \mu_1 = c_2(b_1 - c_1), \mu_2 = b_3(c_1 - b_1),\\ &\lambda_2 = 2c + b_1(a_1 + c_1 - b_1) + c_2(b_2 - c_2) + b_3(c_3 - b_3),\\ &\lambda_3 = 2c + c_1(a_1 + b_1 - c_1) + c_2(b_2 - c_2) + b_3(c_3 - b_3). \end{aligned} \tag{4.3}$$

Since Ric has two different eigenvalues, according to Cayley-Hamilton theorem, we know that $e_i, Ric(e_i), Ric(Ric(e_i)),\ \forall\ i = 1, 2, 3$ are linearly dependent. Hence we get $\mu_1\mu_2 = 0$, i.e. $-c_2 b_3(b_1 - c_1)^2 = 0$. We discuss the following three cases.

Case 1: $b_1 = c_1$**.** In this case, $\lambda_1, \lambda_2, \lambda_3$ are the three eigenvalues of Ric, with $\lambda_2 = \lambda_3$ and $\lambda_1 - \lambda_2 = b_1(a_1 - 2b_1) - c_2(b_2 - c_2) + b_3(b_3 - c_3)$. Since Ric has exactly two different eigenvalues, we have $\lambda_1 - \lambda_2 = b_1(a_1 - 2b_1) - c_2(b_2 - c_2) + b_3(b_3 - c_3) \neq 0$, and e_1 corresponds to the simple eigenvalue of Ric. Since $b_1 = c_1$, from (4.2) we know that e_1, e_2 and e_3 are eigenvectors of A_{Je_1}, with e_2, e_3 belonging to the same eigenspace of A_{Je_1}. Moreover, the second fundamental form has the following form.

$$\begin{aligned} &h(e_1,e_1) = a_1 Je_1,\ h(e_2,e_2) = b_1 Je_1 + b_2 Je_2 + b_3 Je_3,\\ &h(e_1,e_2) = b_1 Je_2,\ h(e_2,e_3) = b_3 Je_2 + c_2 Je_3,\\ &h(e_1,e_3) = b_1 Je_3,\ h(e_3,e_3) = b_1 Je_1 + c_2 Je_2 + c_3 Je_3, \end{aligned} \tag{4.4}$$

where $b_1(a_1 - 2b_1) - c_2(b_2 - c_2) + b_3(b_3 - c_3) \neq 0$.

Case 2: $c_2 = b_3 = 0, b_1 \neq c_1$**.** In this case, $\lambda_1, \lambda_2, \lambda_3$ are the three eigenvalues of Ric. We have that

$$\begin{aligned} &\lambda_1 - \lambda_2 = c_1(a_1 - b_1 - c_1),\\ &\lambda_1 - \lambda_3 = b_1(a_1 - b_1 - c_1),\\ &\lambda_2 - \lambda_3 = (b_1 - c_1)(a_1 - b_1 - c_1). \end{aligned} \tag{4.5}$$

Since Ric has exact two different eigenvalues, we have $a_1 - b_1 - c_1 \neq 0,\ b_1 c_1 = 0$. Without loss of generality, we may assume that $c_1 = 0$. Then $\lambda_1 = \lambda_2 \neq \lambda_3$. Hence e_3 corresponds to the simple eigenvalue of Ric. Since $c_1 = c_2 = b_3 = 0$, from (4.2) we know that e_1, e_2 and e_3 are eigenvectors of A_{Je_3}, with e_1, e_2 belonging to the same eigenspace of A_{Je_3}.

Moreover, the second fundamental form has the following form.

$$\begin{aligned} &h(e_1,e_1)=a_1Je_1,\ h(e_2,e_2)=b_1Je_1+b_2Je_2,\\ &h(e_1,e_2)=b_1Je_2,\ h(e_2,e_3)=0,\\ &h(e_1,e_3)=0,\ h(e_3,e_3)=c_3Je_3, \end{aligned} \tag{4.6}$$

where $b_1\neq 0, a_1\neq b_1$. Hence we conclude like in the previous case that if v spans the 1-dimensional eigenspace of Ric, then v is an eigenvector of A_{Jv}. Moreover on the orthogonal complement of v, A_{Jv} is a multiple of the identity.

Case 3: $c_2b_3=0, c_2^2+b_3^2\neq 0, b_1\neq c_1$. Without loss of generality, we may assume that $c_2=0, b_3\neq 0, b_1\neq c_1$. Then e_2 is an eigenvector of Ric, with the corresponding eigenvalue λ_2. And the other two eigenvalues of Ric are given by

$$\begin{aligned} \eta_1&=\frac{1}{2}(4c+(a_1-b_1)b_1+(2a_1+b_1)c_1-2c_1^2+b_3(-b_3+c_3)-\sqrt{\eta_0}),\\ \eta_3&=\frac{1}{2}(4c+(a_1-b_1)b_1+(2a_1+b_1)c_1-2c_1^2+b_3(-b_3+c_3)+\sqrt{\eta_0}), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} \eta_0=&a_1^2b_1^2+b_1^4+2b_1^3c_1+b_3^2(4c_1^2+(b_3-c_3)^2)+2b_1b_3c_1(-5b_3+c_3)\\ &-2a_1b_1(b_1(b_1+c_1)+b_3(-b_3+c_3))+b_1^2(c_1^2+2b_3(b_3+c_3)). \end{aligned} \tag{4.8}$$

Hence $\eta_1=\eta_3$ if and only if $\eta_0=0$. However, we observe that we can rewrite η_0 in the following way: $\eta_0=(b_1(b_1+c_1-a_1)-b_3(b_3-c_3))^2+4b_3^2(b_1-c_1)^2$. Since in this case we have $b_3\neq 0$ and $b_1\neq c_1$, then $\eta_0>0$, which means that $\eta_1\neq\eta_3$. Since Ric has exactly two different eigenvalues, it follows that either $\lambda_2=\eta_1$ or $\lambda_2=\eta_3$. Hence λ_2 satisfies the equation $(\lambda_2-\eta_1)(\lambda_2-\eta_3)=0$, which is equivalent to

$$\eta:=a_1^2c_1+c_1(-2b_3^2+(b_1+c_1)^2)+b_3c_3(b_1+c_1)+a_1(b_3^2-2c_1(b_1+c_1)-b_3c_3)=0. \tag{4.9}$$

We observe the following relation between η_0 and η:

$$\eta_0=4(b_1-c_1)\eta+((b_1-2c_1)(b_1+c_1-a_1)+b_3(b_3-c_3))^2. \tag{4.10}$$

Hence, under the condition that $\eta=0$, if necessary by changing the order of η_1 and η_3, we can simplify η_1 and η_3.

$$\begin{aligned} \eta_1&=2c+b_1(a_1+c_1-b_1)+b_3(c_3-b_3),\\ \eta_3&=2c+2c_1(a_1-c_1). \end{aligned} \tag{4.11}$$

And the corresponding eigenvectors are given by

$$\tilde e_1=\frac{(b_1+c_1-a_1)e_1+b_3e_3}{\sqrt{b_3^2+(b_1+c_1-a_1)^2}},\tilde e_3=\frac{b_3e_1-(b_1+c_1-a_1)e_3}{\sqrt{b_3^2+(b_1+c_1-a_1)^2}}, \tag{4.12}$$

Since $\eta=0$, we have $\eta_1=\lambda_2\neq\eta_3$, hence $\tilde e_3$ corresponds to the simple eigenvalue of Ric.

Under the orthonormal basis $\{\tilde e_1,e_2,\tilde e_3\}$, the second fundamental form has the following form.

$$\begin{aligned} &h(\tilde e_1,\tilde e_1)=\tilde a_1J\tilde e_1+\tilde b_3J\tilde e_3,\ h(e_2,e_2)=\tilde b_1J\tilde e_1+b_2Je_2+\tilde b_3J\tilde e_3,\\ &h(\tilde e_1,e_2)=\tilde b_1Je_2,\ h(e_2,\tilde e_3)=\tilde b_3Je_2,\\ &h(\tilde e_1,\tilde e_3)=\tilde b_3J\tilde e_1,\ h(\tilde e_3,\tilde e_3)=\tilde c_3J\tilde e_3. \end{aligned} \tag{4.13}$$

So we know that $\tilde e_1,e_2$ and $\tilde e_3$ are eigenvectors of $A_{J\tilde e_3}$, with $\tilde e_1,e_2$ belonging to the same eigenspace of $A_{J\tilde e_3}$, which leads to the same conclusions as in the previous two cases.

Summarizing the above arguments, in all the three cases, if necessary by renaming the basis, we get an orthonormal basis $\{e_1,e_2,e_3\}$ of T_pM, such that e_1,e_2,e_3 are eigenvectors of Ric, with e_1 and e_2 belonging to the 2-dimensional eigenspace and e_3 belonging to the 1-dimensional eigenspace. At the same time, e_1, e_2, e_3 are eigenvectors of A_{Je_3}, with e_1 and e_2 belonging to the same eigenspace of A_{Je_3}. In fact, in case 1, we can rename $\{e_3, e_2, e_1\}$ as $\{e_1, e_2, e_3\}$. In case 2, $\{e_1, e_2, e_3\}$ already satisfies the desired property. In case 3, we can rename $\{\tilde{e}_1, e_2, \tilde{e}_3\}$ as $\{e_1, e_2, e_3\}$.

Finally, in order to get the vector fields $\{E_1, E_2, E_3\}$ in a neighborhood V of the point p, it is sufficient to take E_3 in the 1-dimensional eigenspace which is determined by the simple eigenvalue of the Ricci tensor (it is well known that E_3 is a differentiable vector field), and then choose E_1 and E_2 which are local differentiable vector fields belonging to the 2-dimensional eigenspace of the Ricci tensor. For more details on how to get such vector fields $\{E_1, E_2, E_3\}$, we refer to the book of Kobayashi and Nomizu [19] (page 38) and Lemmas 1.1-1.2 of [20]. In view of the above remarks this implies that at each point of V, E_3 is an eigenvector of A_{JE_3} and that the restriction of A_{JE_3} to the orthogonal complement of E_3 is a multiple of the identity. It is then clear from the symmetry properties of the second fundamental form of a Lagrangian submanifold in $\bar{M}^3(4c)$ that the frame $\{E_1, E_2, E_3\}$ is the required local frame. $\square$

Proof of Theorem 1.2: First, we consider that M^3 is one of the homogeneous 3-manifolds (the Berger spheres, the Heisenberg group Nil_3 and the universal covering of the Lie group $\mathrm{PSL}(2,\mathbb{R})$). From the properties of these homogeneous 3-manifolds described in Section 3, there exists a canonical frame $\{E_1, E_2, E_3\}$, with respect to which, the Levi-Civita connections are given by

$$\begin{aligned}
&\nabla_{E_i}E_i = 0,\ i = 1,2,3,\\
&\nabla_{E_1}E_2 = -\tau E_3,\ \nabla_{E_1}E_3 = \tau E_2,\\
&\nabla_{E_2}E_1 = \tau E_3,\ \nabla_{E_2}E_3 = -\tau E_1,\\
&\nabla_{E_3}E_1 = (\tau - \frac{\kappa}{2\tau})E_2,\ \nabla_{E_3}E_2 = (\frac{\kappa}{2\tau} - \tau)E_1,
\end{aligned} \tag{4.14}$$

where $\tau \neq 0$ and $\kappa \neq 4\tau^2$.

From Section 3, we know that the Ricci tensor of M^3 has exactly two different eigenvalues, E_1, E_2, E_3 are eigenvectors of the Ricci tensor, with E_3 corresponding to the simple eigenvalue. Hence, using Proposition 4.1, the second fundamental form of M^3 in $\bar{M}^3(4c)$, $c \in \{-1, 0, 1\}$ has the following form.

$$\begin{cases}
h(E_1,E_1) = a_1JE_1 + a_2JE_2 + b_3JE_3,\ h(E_2,E_2) = b_1JE_1 + b_2JE_2 + b_3JE_3,\\
h(E_1,E_2) = a_2JE_1 + b_1JE_2,\ h(E_2,E_3) = b_3JE_2,\\
h(E_1,E_3) = b_3JE_1,\ h(E_3,E_3) = c_3JE_3.
\end{cases} \tag{4.15}$$

From the Codazzi equation $(\nabla h)(E_1,E_2,E_3) = (\nabla h)(E_2,E_1,E_3)$ we have

$$(a_1 + b_1)\tau + E_2(b_3) = 0,\ (a_2 + b_2)\tau - E_1(b_3) = 0,\ c_3 = 2b_3. \tag{4.16}$$

From $(\nabla h)(E_1,E_3,E_3) = (\nabla h)(E_3,E_1,E_3)$ we have

$$E_3(b_3) = E_1(c_3) = 0. \tag{4.17}$$

From $(\nabla h)(E_2,E_3,E_3) = (\nabla h)(E_3,E_2,E_3)$ we have

$$E_2(c_3) = 0. \tag{4.18}$$

From $(\nabla h)(E_1,E_2,E_1) = (\nabla h)(E_2,E_1,E_1)$ we have

$$4b_3\tau - E_2(a_1) + E_1(a_2) = 0. \tag{4.19}$$

From $(\nabla h)(E_1, E_2, E_2) = (\nabla h)(E_2, E_1, E_2)$ we have
$$4b_3\tau - E_2(b_1) + E_1(b_2) = 0. \tag{4.20}$$
Combining the above equations, we immediately get that
$$b_1 = -a_1, b_2 = -a_2, b_3 = c_3 = 0. \tag{4.21}$$

The above relations show that ϕ is minimal in this case.

Using (4.21), from $(\nabla h)(E_1, E_2, E_1) = (\nabla h)(E_2, E_1, E_1)$ we have
$$E_2(a_1) = E_1(a_2),\ E_1(a_1) = -E_2(a_2). \tag{4.22}$$
From $(\nabla h)(E_1, E_3, E_1) = (\nabla h)(E_3, E_1, E_1)$ we have
$$E_3(a_1) = \frac{4\tau^2 - 3\kappa}{2\tau}a_2,\ E_3(a_2) = \frac{3\kappa - 4\tau^2}{2\tau}a_1. \tag{4.23}$$

Using the Gauss equation (2.2), we have
$$\begin{aligned} \langle R(E_1, E_2)E_2, E_1\rangle &= \kappa - 3\tau^2 = c - 2(a_1^2 + a_2^2), \\ \langle R(E_1, E_3)E_3, E_1\rangle &= \tau^2 = c. \end{aligned} \tag{4.24}$$
Hence, we get $c = \tau^2 > 0$, then $c = 1$ and $\tau^2 = 1$.

Moreover, we get that $a_1^2 + a_2^2 = 2 - \frac{\kappa}{2}$ is constant, which means that $E_i(a_1^2 + a_2^2) = 0,\ \forall\, i = 1, 2, 3$. Then using (4.22) we obtain that $E_1(a_1) = E_2(a_1) = E_1(a_2) = E_2(a_2) = 0$.

Next it follows from $[E_1, E_2] = -2\tau E_3$ and $E_1(a_1) = E_2(a_1) = E_1(a_2) = E_2(a_2) = 0$ that $E_3(a_1) = E_3(a_2) = 0$. So we get that both a_1 and a_2 are constants. This together with (4.23) implies that $\kappa = \frac{4}{3} > 0$ and $a_1^2 + a_2^2 = 2 - \frac{\kappa}{2} = \frac{4}{3}$. Hence we get that M^3 is the Berger sphere $\mathbb{S}_b^3(\frac{4}{3}, 1)$. Moreover, since a_1 and a_2 are constant, we can rotate E_1 and E_2 in the subspace which is orthogonal to E_3, to make $a_2 = 0$ and $a_1 = \frac{2}{\sqrt{3}}$. Then using the uniqueness theorem, we obtain that the immersion is congruent with the immersion given in Section 3.

Second, we consider that M^3 is the Lie group Sol_3. We use the frame given in Section 3, together with the expression of the second fundamental form. After an analogous argument to that for the homogeneous 3-manifolds (the Berger spheres, the Heisenberg group Nil_3 and the universal covering of the Lie group $\mathrm{PSL}(2, \mathbb{R})$), using Codazzi equations we deduce that $a_2 = b_1 = b_3 = c_3 = 0$. It then follows from the Gauss equations that $1 - c = 0$ and $1 + c = 0$, which is a contradiction.

This completes the proof of Theorem 1.2. □

If the immersion is moreover minimal, using Proposition 4.1 together with its proof, we get the following corollary.

Corollary 4.2 (c.f. [11])**.** *Let M^3 be a minimal Lagrangian quasi-Einstein submanifold in a complex space form $\tilde{M}^3(4c)$, $c \in \{-1, 0, 1\}$. Let $p \in M$ be a non-totally geodesic point. Then either M^3 is Einstein, or there exists local orthonormal vector fields $\{E_1, E_2, E_3\}$, local positive function λ and local function a such that either*
$$\begin{aligned} (i)\ & h(E_1, E_1) = \lambda JE_1,\ h(E_2, E_2) = -\lambda JE_1, \\ & h(E_1, E_2) = -\lambda JE_2,\ h(E_2, E_3) = 0, \\ & h(E_1, E_3) = 0,\ h(E_3, E_3) = 0. \end{aligned} \tag{4.25}$$

or

$$\begin{aligned}(ii)\ & h(E_1,E_1)=2\lambda JE_1,\ h(E_2,E_2)=-\lambda JE_1+aJE_2,\\ & h(E_1,E_2)=-\lambda JE_2,\ h(E_2,E_3)=-aJE_3,\\ & h(E_1,E_3)=-\lambda JE_3,\ h(E_3,E_3)=-\lambda JE_1-aJE_2.\end{aligned} \tag{4.26}$$

Using Corollary 4.2, we prove the following classification theorem for minimal Lagrangian quasi-Einstein submanifolds in a complex space form $\bar{M}^3(4c)$, $c\in\{-1,0,1\}$.

Theorem 4.3 (Same with Theorem 1.4)**.** *Let M^3 be a minimal Lagrangian quasi-Einstein submanifold in a complex space form $\bar{M}^3(4c)$, $c\in\{-1,0,1\}$. Assume that M^3 has constant scalar curvature, then locally M^3 is congruent to one of the following*

- *M^3 is totally geodesic.*
- *M^3 is a flat torus in $\mathbb{CP}^3$.*
- *M^3 is the Berger sphere $\mathbb{S}^3_b(\frac{4}{3},1)$ with the immersion given by $\phi:\mathbb{S}^3_b(\frac{4}{3},1)\to\mathbb{CP}^3(4)$, $(z,w)\mapsto[\bar z^3+3\bar z\bar w^2,\sqrt{3}(\bar z^2w+\bar w|w|^2-2\bar w|z|^2),\sqrt{3}(\bar zw^2+z|z|^2-2z|w|^2),w^3+3z^2w]$.*
- *M^3 is locally isometric with $\mathbb{S}^2\times\mathbb{R}$, $\bar{M}^3(4c)=\mathbb{CP}^3$, and the immersion is obtained as the Calabi product of the totally geodesic immersion of $\mathbb{S}^2$ into $\mathbb{CP}^2$ and a point.*

Proof. We will explore the Codazzi equations (2.3) and Gauss equations (2.2) to determine the submanifold M^3, by using Corollary 4.2,

Case 1: M^3 is Einstein.

In this case, M^3 has constant sectional curvature, then it follows from Theorem 1.1 (see also [16]) that either M^3 is totally geodesic or M^3 is a flat torus in the complex projective space $\mathbb{CP}^3$.

Case 2: The second fundamental form of M^3 has the form of (i) in Corollary 4.2.

In this case, since M^3 has constant scalar curvature, we can deduce directly from Codazzi equations and Gauss equations that $\lambda^2=\frac{4c}{3}$. Hence we get $c=1$ and $\lambda=\frac{2}{\sqrt{3}}$. Then the second fundamental form of M^3 has the same form with the example given in Section 3. The immersion moreover satisfies Chen's equality and has constant scalar curvature. Applying now the classification result of [6] we deduce that M^3 is locally isometric with the Berger sphere $\mathbb{S}^3_b(\frac{4}{3},1)$ and the immersion is unique. Then using the analytic expression obtained in [18] or [14], we get that the immersion is locally congruent to the immersion $\phi:\mathbb{S}^3_b(\frac{4}{3},1)\to\mathbb{CP}^3(4)$, $(z,w)\mapsto[\bar z^3+3\bar z\bar w^2,\sqrt{3}(\bar z^2w+\bar w|w|^2-2\bar w|z|^2),\sqrt{3}(\bar zw^2+z|z|^2-2z|w|^2),w^3+3z^2w]$.

In the following two cases, we assume that M^3 is not Einstein, we introduce local functions $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3$ by

$$\begin{aligned}&\nabla_{E_1}E_1=x_1E_2+x_2E_3,\ \nabla_{E_1}E_2=-x_1E_1+x_3E_3,\ \nabla_{E_1}E_3=-x_2E_1-x_3E_2,\\ &\nabla_{E_2}E_1=y_1E_2+y_2E_3,\ \nabla_{E_2}E_2=-y_1E_1+y_3E_3,\ \nabla_{E_2}E_3=-y_2E_1-y_3E_2,\\ &\nabla_{E_3}E_1=z_1E_2+z_2E_3,\ \nabla_{E_3}E_2=-z_1E_1+z_3E_3,\ \nabla_{E_3}E_3=-z_2E_1-z_3E_2.\end{aligned} \tag{4.27}$$

Case 3: The second fundamental form of M^3 has the form of (ii) in Corollary 4.2 with $a=0$.

In this case, from $(\nabla h)(E_1,E_2,E_3)=(\nabla h)(E_2,E_3,E_1)=(\nabla h)(E_3,E_1,E_2)$ we get that $E_2(\lambda)=0=x_1\lambda$ and $E_3(\lambda)=0=x_2\lambda$, hence $x_1=x_2=0$.

From $(\nabla h)(E_1, E_2, E_1) = (\nabla h)(E_2, E_1, E_1)$ we get that $E_1(\lambda) = -4y_1\lambda$ and $y_2\lambda = 0$, hence $y_2 = 0$.

From $(\nabla h)(E_2, E_3, E_3) = (\nabla h)(E_3, E_2, E_3)$ we get that $y_1\lambda = z_2\lambda$ and $3y_2\lambda = z_1\lambda$, hence $z_1 = y_2 = 0$ and $y_1 = z_2$.

Using Corollary 4.2, we know that the scalar curvature of M^3 is $R = 6c - 10\lambda^2$. Since M^3 has constant scalar curvature, we get that λ is constant, which leads to $E_1(\lambda) = 0$. Then from the information above we have that $y_1 = z_2 = 0$. It then follows immediately that M^3 has parallel second fundamental form. Moreover, from Gauss equations, we have that $\lambda^2 = \frac{c}{3}$. Then we get $c = 1$ and $\lambda^2 = \frac{1}{3}$.

Hence according to Theorem 1.6 in [17] (cf.[12]), the immersion is a Calabi product of a 2-dimensional minimal parallel Lagrangian submanifold in $\mathbb{CP}^2$ and a point. Note that a parallel surface has constant Gaussian curvature. Applying Ejiri's classification result ([13]) gives that the surface is either flat (in which case M^3 is flat as well, which we excluded) or totally geodesic. This means that M^3 is locally isometric with $\mathbb{S}^2 \times \mathbb{R}$, and the immersion is obtained as the Calabi product of the totally geodesic immersion of $\mathbb{S}^2$ into $\mathbb{CP}^2$ and a point.

Case 4: The second fundamental form of M^3 has the form of (ii) in Corollary 4.2 with $a \neq 0$.

In this case, from $(\nabla h)(E_1, E_2, E_1) = (\nabla h)(E_2, E_1, E_1)$ we get that

$$\begin{aligned} &E_2(\lambda) = 2x_1\lambda, \\ &E_1(\lambda) = -4y_1\lambda - x_1 a, \\ &4y_2\lambda = x_2 a. \end{aligned} \tag{4.28}$$

From $(\nabla h)(E_1, E_2, E_3) = (\nabla h)(E_2, E_3, E_1) = (\nabla h)(E_3, E_1, E_2)$ we get that

$$\begin{aligned} &x_2 a = 4y_2\lambda = 4z_1\lambda, \\ &3x_3 a - x_2\lambda = y_2 a = -E_3(\lambda) - c_{!}a, \\ &-E_1(a) - x_1\lambda = -E_2(\lambda) + y_1 a = z_2 a. \end{aligned} \tag{4.29}$$

From $(\nabla h)(E_1, E_3, E_1) = (\nabla h)(E_3, E_1, E_1)$ we get that

$$E_3(\lambda) = 2x_1\lambda. \tag{4.30}$$

From $(\nabla h)(E_1, E_2, E_2) = (\nabla h)(E_2, E_1, E_2)$ we get that

$$E_2(\lambda) = -E_1(a) + 3x_1\lambda - y_1 a. \tag{4.31}$$

From $(\nabla h)(E_2, E_3, E_3) = (\nabla h)(E_3, E_2, E_3)$ we get that

$$\begin{aligned} E_2(a) &= -y_1\lambda + z_2\lambda - 3z_3 a, \\ E_3(a) &= -z_1\lambda + 3y_2\lambda + 3y_3 a. \end{aligned} \tag{4.32}$$

By analyzing the equations above, we get that

$$\begin{aligned} &x_1 = x_2 = x_3 = y_2 = z_1 = 0, \ y_1 = z_2, \\ &E_1(\lambda) = -4y_1\lambda = -4z_2\lambda, \\ &E_2(\lambda) = E_3(\lambda) = 0, \\ &E_1(a) = -y_1 a, \ E_2(a) = -3z_3 a, \ E_3(a) = 3y_3 a. \end{aligned} \tag{4.33}$$

From Corollary 4.2, we know that the scalar curvature of M^3 is $R = 6c - 4a^2 - 10\lambda^2$. Since M^3 has constant scalar curvature, we have that $E_i(R) = 0, \ \forall \ i = 1, 2, 3$, using (4.33), we immediately obtain that $y_1 = z_3 = y_3 = 0$. Then it follows that both λ and a

are constant. Moreover, it follows from Gauss equations that $\lambda^2 = \frac{c}{3}$ and $a^2 = \frac{2c}{3}$. Hence, we have in this case that $c = 1$ and M^3 is flat which we excluded.

Therefore, we complete the proof of Theorem 4.3. $\square$

Proof of Theorem 1.6 and Theorem 1.7: Note that both $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are quasi-Einstein and have constant scalar curvatures. Since in Theorem 1.6 and Theorem 1.7, we assume that the immersion is minimal, we obtain Theorem 1.6 and Theorem 1.7 immediately from Theorem 4.3.

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